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# Thermal Entanglement of XXZ Heisenberg Chain under Rectangle Magnetic Field

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**Abstract** We study thermal entanglement of $XXZ$ Heisenberg chain under rectangle magnetic field. Under this magnetic field $B$, the region of thermal entanglement in terms of $B$ and temperature $T$ can be extended. Moreover, one can improve threshold temperature, where entanglement vanish, just by increasing the strength of magnetic field. This effect is similar to that of the anisotropic coupling of spin in $XY$ plane but provide us a realizable method to improve threshold temperature.

## 1 Introduction

Entanglement as nonlocal correlations between quantum systems [1, 2] has been recognized as crucial role in quantum information such as quantum cryptography, quantum teleportation and quantum computation. Thermal entanglement in Heisenberg chain has been studied widely [3–12] because Heisenberg model is simple but realizable. Moreover, Heisenberg model can be realized not only in solid state system but also in quantum dots [7], nuclear spins [13], cavity QED [14, 15]. References [4, 8] find that the anisotropy in Heisenberg model can be used to control the extent of entanglement and to improve threshold temperature at which entanglement vanish. However, anisotropy is intestine property of a chain and difficult to control. Reference [10] studied effects of inhomogeneous magnetic field while Ref. [11] investigated impurity of magnetic field on the entanglement.

In this paper, we introduce a rectangle magnetic field to a Heisenberg $XXZ$ chain. After studying a two-qubit and multi-qubit, we find that a rectangle magnetic field can extend the regions of entanglement existence. We can improve the threshold temperature just by increasing the magnetic field. This property is similar to that of the anisotropic $XY$ Heisenberg chain. However, the anisotropic parameters are usually fixed when the chain is made. By exerting an external rectangle magnetic field, we could easy control the entanglement and improve the threshold temperature.

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## 2 The Model

Firstly, we introduce a kind of magnetic field which is showed in Fig. 1. For the odd spins the magnetic fields are exerted while for the even spins no magnetic fields are exerted. This form of fields is looked like a rectangle wave. Hereafter, we call this kind of magnetic fields as rectangle magnetic fields.

Under our magnetic fields, the $N$-qubit Hamiltonian of a Heisenberg spin chain can be written as

$$H_1 = \frac{1}{2}\sum_{i=1}^{N}(J_x\sigma_i^x\sigma_{i+1}^x + J_y\sigma_i^y\sigma_{i+1}^y + J_z\sigma_i^z\sigma_{i+1}^z + B_i\sigma_i^z) \tag{1}$$

with

$$B_i = \begin{cases} B & \text{for } i = \text{odd number},\\ 0 & \text{for } i = \text{even number},\end{cases} \tag{2}$$

where $\sigma_i$ $(i = x, y, z)$ are Pauli matrices and $B$ is the strength of magnetic field. $J_x$, $J_y$, $J_z$ are the real coupling coefficients. If an isotropic chain in $XY$ plane, $J_x = J_y = J$. In order to see the effect of rectangle field, we will study $XXZ$ chain which means the isotropy in $XY$ plane.

For a system in equilibrium at temperature $T$, the density operator is $\rho = \frac{1}{Z}\exp(-H/\kappa_B T)$, where $Z = \mathrm{Tr}[\exp(-H/\kappa_B T)]$ is the partition function and $\kappa_B$ is the Boltzmann constant. For simplicity, we write $\kappa_B = 1$. We recall concurrence $C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$ [16] which can be used to measure the thermal entanglement where $\lambda_i$ are the square roots of the eigenvalues of $R$ in decreasing order and the matrix $R = \rho^r s\rho^{r*}s$ $(s = \sigma^y \otimes \sigma^y)$, where $\rho^r$ is the reduced density matrix of those two qubits by tracing out the state of the other qubits from $\rho$.

For two-qubit $XXZ$ Heisenberg chain, the Hamiltonian is

$$H_1 = J(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) + \frac{J_z}{2}\sigma_1^z\sigma_2^z + \frac{B}{2}\sigma_1^z \tag{3}$$

where $J = \frac{J_x+J_y}{2}$.

On the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, we deduce the square roots $\lambda_i$ of matrix $R$ as

$$\begin{aligned}\lambda_{1,2} &= \frac{e^{-\frac{J_z}{2T}}}{Z},\\ \lambda_{3,4} &= \frac{e^{\frac{J_z}{2T}}}{Z}\left[\sqrt{1+\left(\frac{J}{\xi}\sinh\frac{\xi}{T}\right)^2} \pm \frac{J}{\xi}\sinh\frac{\xi}{T}\right],\end{aligned} \tag{4}$$

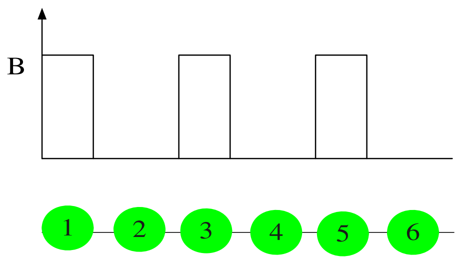


**Fig. 1** The illustration of rectangle magnetic field

and the partition function

$$Z = 2e^{-\frac{J_z}{2T}} \cosh \frac{B}{2T} + 2e^{\frac{J_z}{2T}} \cosh \frac{\xi}{T} \tag{5}$$

with $\xi = \sqrt{(\frac{B}{2})^2 + J^2}$. One can easy check that $\lambda_3$ (with +) is the largest one; therefore the Concurrence $C_1 = \max\{0, \frac{2}{Z}(e^{\frac{J_z}{2T}} \frac{J}{\xi} \sinh \frac{\xi}{T} - e^{-\frac{J_z}{2T}})\}$.

In order to see the effect of the rectangle magnetic fields, we will compare it with an anisotropic $XYZ$ Heisenberg chain under homogeneous magnetic field. The Hamiltonian for the two-qubit anisotropic $XYZ$ Heisenberg chain is as

$$\begin{aligned} H_2 = {} & J(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+) + J\gamma(\sigma_1^+\sigma_2^+ + \sigma_1^-\sigma_2^-) \\ & + \frac{J_z}{2}\sigma_1^z\sigma_2^z + \frac{B}{2}(\sigma_1^z + \sigma_2^z), \end{aligned} \tag{6}$$

where $J = \frac{J_x+J_y}{2}$, $\gamma = \frac{J_x-J_y}{J_x+J_y}$. The square roots of matrix $R$ are

$$\begin{aligned} \lambda'_{1,2} &= \frac{e^{\frac{J_z}{2T} \pm \frac{J}{T}}}{Z'}, \\ \lambda'_{3,4} &= \frac{e^{-\frac{J_z}{2T}}}{Z'}\left[\sqrt{1 + \left(\frac{J\gamma}{\eta} \sinh \frac{\eta}{T}\right)^2} \pm \frac{J\gamma}{\eta} \sinh \frac{\eta}{T}\right] \end{aligned} \tag{7}$$

and

$$\begin{aligned} Z' &= 2\left(e^{-\frac{J_z}{2T}} \cosh \frac{\eta}{T} + e^{\frac{J_z}{2T}} \cosh \frac{J}{T}\right), \\ \eta &= \sqrt{B^2 + (J\gamma)^2}. \end{aligned} \tag{8}$$

It is a little bit difficult to direct judge the quantity of $\lambda'_1$. So, the Concurrence $C_2 = \max\{0, 2\max(\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4) - \sum_i \lambda'_i\}$. For showing our magnetic fields effect clearly, we will plot the difference between $C_1$ and $C_2$

$$\Delta C = C_1 - C_2. \tag{9}$$

Comparing $\lambda_{3,4}$ with $\lambda'_{3,4}$, we notice that only when the anisotropic parameter $\gamma \neq 0$, the result $\lambda'_{3,4}$ have the similar form to that of $\lambda_{3,4}$. We will show the effect of the similar form.

## 3 The Effects of Rectangle Magnetic Field on Entanglement

Figure 2(a) shows the entanglement as a function of magnetic field and temperature when two-qubit are under rectangle magnetic field. One can see clearly that the threshold temperature, where entanglement drop to zero, can be increased with increasing of magnetic field within the whole region. However, the phenomenon, threshold temperature increasing with increasing of $B$, only can be observed in "revival region" in anisotropic $XY$ ($XYZ$) chain [8]. Usually, the coupling constants $J_i$ ($i = x, y, z$) are difficult to adjust, which means employing the anisotropy $\gamma$ in $XY$ plane to enhance threshold temperature face difficulty. Instead of using the internal parameter $\gamma$, here we can employ the external rectangle magnetic field to realize high threshold temperature just by increasing the strength of magnetic

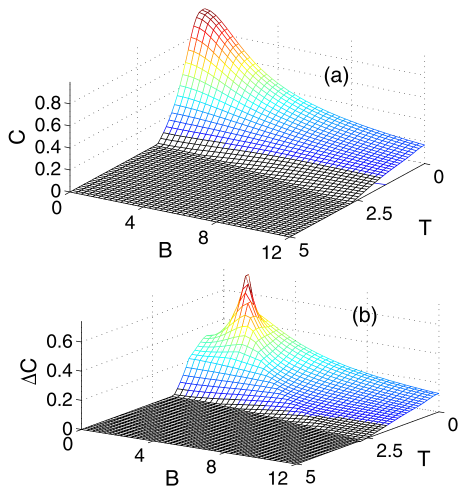


**Fig. 2** **a** The entanglement as a function of magnetic field $B$ and temperature $T$ for two-qubit under rectangle magnetic field. **b** Entanglement difference between XXZ chain under our rectangle magnetic field and XYZ chain under homogeneous magnetic field where $\gamma = 0.5$. For *both of these plot*, $J = J_z = 1$

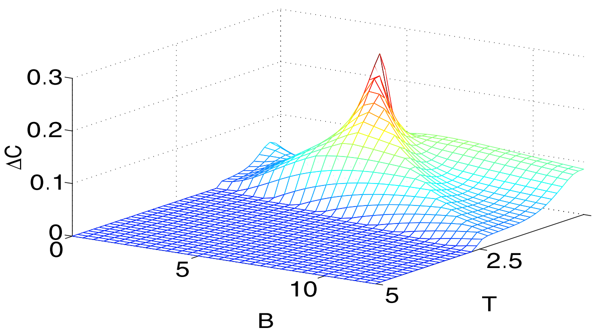


**Fig. 3** The entanglement difference $\Delta C$ between isotropy case under rectangle magnetic field and anisotropy case under homogeneous magnetic field, where $\gamma = 0.5$, $J = J_z = 1$

field. The inhomogeneous in magnetic field, at some degree, has the similar effect to the anisotropy. We can understand it from the expression of (4) and (7). Only when $\gamma \neq 0$, $\lambda'_{3,4}$ have the similar form to $\lambda_{3,4}$. So, anisotropy for homogeneous magnetic field is the condition to exhibit threshold temperature increasing with $B$. But for a rectangle magnetic field, $\lambda_{3,4}$ keep its similar form no any other condition except existence of coupling $J$. Therefore, rectangle magnetic field can automatically provide us a property of anisotropy chain. Moreover, the amount of entanglement under rectangle magnetic field is larger than that under homogeneous field, which is shown in Fig. 2(b).

Rectangle magnetic field should have its exact meaning only when it is applied in multi-qubit. Now, we extend this magnetic field to six qubits system. We numerical calculate the entanglement difference $\Delta C$ between our case from (1) and that of the extension for six qubits from (6). Obviously, for the parameter $\gamma = 0.5$, $J = J_z = 1$, pairwise entanglement of isotropy $XXX$ chain is larger than that of anisotropy $XYZ$ chain. Besides intrinsic advantage with rectangle magnetic fields, for multi-qubit case, $\Delta C > 0$ has another explanation. Usually, when we study multi-qubit, the entanglement will decrease [3], which can be explained as the pair of qubits interacting with its "environment" other spin qubits. If the magnetic field is homogeneous, every qubit will be exerted the same amount of magnetic field. But when we apply rectangle magnetic field, only the spins with odd number will be in magnetic field. This field equal to partial "isolate" the pair qubits from the interaction with

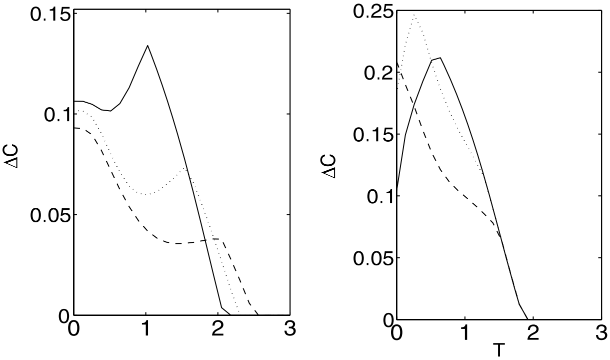


**Fig. 4** The entanglement difference between isotropy case under rectangle magnetic field and anisotropy case under homogeneous magnetic field, where (**a**) $B = 6$ (*solid line*), $B = 8$ (*dotted line*), $B = 10$ (*dashed line*). $\gamma = 0.5$, $J = J_z = 1$. (**b**) $\gamma = 0.9$ (*solid line*), $\gamma = 0.6$ (*dotted line*), $\gamma = 0.3$ (*dashed line*); other parameter are $B = 4$, $J = J_z = 1$

its "environment". Therefore, for multi-qubit, entanglement with rectangle magnetic field will drop less than that with equal magnetic field.

Figure 4 shows us the entanglement difference $\Delta C$ changes with temperature for several value of $B$ and $\gamma$. Although at low temperature the larger $B$ do not bring larger $\Delta C$, the threshold temperature, at which entanglement vanish, really increase with increasing of $B$ shown in Fig. 4(a). Figure 4(b) shows us the $\Delta C(T)$ for several value of anisotropy $\gamma$. Because different $\gamma$ make the revival region different, $\Delta C$ differ at low temperature region for different $\gamma$. But for higher temperature region, three curves merge. This is because the entanglement within this region has dropped to zero for anisotropic $XYZ$ chain ($C_2 = 0$) so that $\Delta C = C_1$, which show us again the rectangle magnetic field make the entanglement region extend.

## 4 Conclusion

We study thermal entanglement when a $XXZ$ Heisenberg chain is exerted a rectangle magnetic field for two-qubit and multi-qubit cases. It is found that entanglement region in terms of $B$ and $T$ can be extended. And the threshold temperature can be improved just by increasing the external magnetic field. Comparing with the case of employing anisotropy in XY plane to increase threshold temperature, our method give a external adjustable parameter, that is to say, using rectangle magnetic field is realizable in experiment.

## References

1. Bennett, C.H., DiVincenzo, D.P.: Nature **404**, 247 (2000)
2. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000)
3. Arnesen, M.C., Bose, S., Vedral, V.: Phys. Rev. Lett. **87**, 017901 (2001)
4. Wang, X.: Phys. Rev. A **64**, 012313 (2001)
5. Kamta, G.L., Starace, A.F.: Phys. Rev. Lett. **88**, 107901 (2002)
6. OConnor, K.M., Wootters, W.K.: Phys. Rev. A **63**, 052302 (2001)
7. Loss, D., DiVincenzo, D.P.: Phys. Rev. A **57**, 120 (1998); Burkard, G., Loss, D., DiVincenzo, D.P.: Phys. Rev. B **59**, 2070 (1999)
8. Kamta, G.L., Starace, A.F.: Phys. Rev. Lett. **88**, 107901 (2002)
9. Zhou, L., Song, H.S., Guo, Y.Q., Li, C.: Phys. Rev. A **68**, 024301 (2003)
10. Zhang, G.F., Li, S.S.: Phys. Rev. A **72**, 034302 (2005)
11. Fu, H., Solomon, A.I., Wang, X.: J. Phys. A **35**, 4293 (2002)

12. Yang, G.H., Gao, W.B., Zhou, L., Song, H.S.: quant-ph/0602051
13. Kane, B.E.: Nature **393**, 133 (1998)
14. Imamoglu, A., Awschalom, D.D., Burkard, G., DiVincenzo, D.P., Loss, D., Sherwin, M., Small, A.: Phys. Rev. Lett. **83**, 4204 (1999)
15. Zheng, S.B., Guo, G.C.: Phys. Rev. Lett. **85**, 2392 (2000)
16. OConnor, K.M., Wootters, W.K.: Phys. Rev. A **63**, 052302 (2001)